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SUPERSTRING AMPLITUDES AND ELEMENTARY PARTICLES

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Abstract

It has been shown recently that superstring states are organized into supermultiplets for the massless fundamental modes and solitons representing excited states. Summation of superstring amplitudes yields an expression that has poles when the momenta of the external states have specific values. It is established that only the residues and not the mass can be shifted by higher genus terms in the superstring perturbation series. The masses of elementary particles will not be generated by radiative corrections in superstring theory. Quantization of the energy levels does occur for the excited modes and the solitons. The superstring must be replaced by a hadronic string to achieve a phenomenologically viable theory of the particle spectrum. A linear spacing for the masses can be derived from the pion bubble model. The vibrational modes of the string between the quark and anti-quark in the pion then generates square-root variations from the integer multiple rule for the excited states of hadrons.

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1. Introduction

While solitons described by extreme black hole geometries might be used to representing elementary particles, excited states generally have a Planck mass, and it is useful to restrict consideration to the massless sector. The feasibility of such a truncation is raised by connections between Type IIA string theory and eleven-dimensional supergravity [23] [24].

An alternative approach of the derivation of the elementary particle spectrum would follow from characteristics of superstring amplitudes with massless external states. It may be recalled that the energy levels of string states are defined by a Fourier mode expansion and expectation values of of operators satisfying a Virasoro algebra. A quantization of energy is introduced. The superstring amplitudes will have poles at the on-shell masses of excited modes defined by the external vertex operators. It is analogous to the poles of the Koba-Nielsen amplitudes in bosonic string scattering. Corrections to the masses at sufficiently high order are prevented, however, by nonrenormalization theorems. Given the quantization of on-shell energies, it would follow that there is a theoretical basis for the integer-multiple rule for the elementary particle spectrum in a superstring model.

An evaluation of resonance poles in superstring correlation functions yields a series of corrections to the residues without affecting the masses. The absence of a mass renormalization would be expected from the finiteness of the superstring perturbation series at each order and summed over the genus for a range of the string coupling. Therefore, higher-genus terms cannot shift the massless states to the masses of elementary particles.

The predicted values of the masses based on the excited states of the superstring would not be considered to be phenomenologically viable. It is necessary to replace the superstring by a hadronic string. If the hadronic string model has supersymmetry, this symmetry would be present in the spectrum [10], which may be valid for two multiplets with a tetraquark resonance. By contrast, a nonsupersymmetric hadronic string model can generate the known spectrum, but the quantum theory has problems of consistency similar to that of the bosonic string.

The integer multiple rule, nevertheless, characterizes a branch of both the meson and baryon resonances. An explanation through pion bubble diagrams [11] is compatible with validity of the rule for the γ branch. This method then may be combined with the hadronic string model to derive a realistic spectrum and a square root variation from the integer multiple rule for the excited states.

2. Scattering Amplitudes

The poles of string amplitudes have integer spacings. The scattering amplitude with N vertex operators would be

$$(1) \qquad \sum_g \kappa_{str}^{2g-2} \int_{M_g} d\mu_{WP} \int_{\Sigma_g} \prod_{i=1}^{n} d^2 z_i \sqrt{g(z_i)} \langle V_1(z_1) ... V_N(z_N), \rangle.$$

where $V_i(z_i)$ is the i^{th} vertex operator, $d\mu_{WP}$ is the Weil-Petersson measure on the moduli space M_g of complex metrics on a Riemann surface of genus g and κ_{str} is the string coupling. The path integral of the time ordered product would be evaluated through the Wick contractions of the operators, which yields

(2)

$$\left\langle 0 \right| \left(1 + ip_1 \cdot X(z_1) - \frac{1}{2!}(p_1 \cdot X(z_1))^2 - \frac{i}{3!}(p_1 \cdot X(z_1))^2 + ... \right)$$

$$\left(1 + ip_2 \cdot X(z_2) - \frac{1}{2!}(p_2 \cdot X(z_2))^2 - \frac{i}{3!}(p_2 \cdot X(z_2))^3 +. .. \right)$$

$$\left(1 + ip_N \cdot X(z_N) - \frac{1}{2!}(p_N \cdot X(z_N))^2 - \frac{i}{3!}(p_N \cdot X(z_n))^3 + ... \right) \left| 0 \right\rangle$$

$$= \langle 0|0 \rangle + i \sum_i \langle 0| p_i \cdot X(z_i) |0\rangle - \sum_{i<j} \langle 0| p_i^\mu p_j^\nu X_\mu(z_i) X_\nu(z_j) |0\rangle$$

$$+ \sum_{i<j<k<\ell} \langle 0| p_i^\mu p_j^\nu X_\mu(z_i) X_\nu(z_j) |0\rangle \langle 0| p_k^\rho p_\ell^\sigma X_\rho(z_k) X_\sigma(z_\ell) |0\rangle$$

$$+ \frac{1}{(2!)^2} \cdot 2 \sum_{i<j} p_i^\mu p_j^\nu p_j^\rho p_j^\sigma \langle 0| X_\mu(z_i) X_\rho(z_j) |0\rangle \langle 0| X_\nu(z_i) X_\sigma(z_j) |0\rangle$$

$$+ \sum_{i<j<k} p_i^\mu p_j^\nu p_j^\rho p_k^\sigma \langle 0| X_\mu(z_i) X_\nu(z_j) |0\rangle \langle 0| X_\rho(z_j) X_\sigma(z_k) |0\rangle$$

$$- \frac{1}{(3!)^2} 6 \sum_{i<j} p_i^\mu p_i^\nu p_i^\lambda p_j^\rho p_j^\sigma p_j^\tau \langle 0| X_\mu(z_i) X_\rho(z_j) |0\rangle \langle 0| X_\nu(z_i) X_\sigma(z_j) |0\rangle \langle 0| X_\lambda(z_i) X_\tau(z_j) |0\rangle$$

$$- \frac{1}{(2!)^3} 2^2 \sum_{i<j<k} p_i^\mu p_i^\nu p_j^\lambda p_j^\rho p_k^\sigma p_k^\tau \langle 0| X_\mu(z_i) X_\lambda(z_j) |0\rangle \langle 0| X_\nu(z_i) X_\sigma(z_k) |0\rangle$$

$$\langle 0| X_\rho(z_j) X_\tau(z_k) |0\rangle$$

(3)

$$-\frac{1}{(2!)^2}2\sum_{i<j<k<\ell} p_i^\mu p_j^\nu p_j^\lambda p_k^\rho p_k^\sigma p_\ell^\tau \Big[\langle 0|X_\mu(z_i)X_\nu(z_j)|0\rangle\langle 0|0X_\lambda(z_j)X_\sigma(z_k)|0\rangle$$

$$\langle 0|X_\rho(z_k)X_\tau(z_\ell)|0\rangle$$

$$\cdots\cdots\cdots\cdots + \langle 0|X_\mu(z_i)X_\rho(z_k)|0\rangle\langle 0|X_\nu(z_j)X_\sigma(z_k)|0\rangle\langle 0|X_\lambda(z_j)X_\tau(z_\ell)|0\rangle$$

$$+ \langle 0|X_\mu(z_i)X_\tau(z_\ell)|0\rangle\langle 0|X_\nu(z_j)X_\rho(z_k)|0\rangle\langle 0|X_\lambda(z_j)X_\sigma(z_k)|0\rangle\Big]$$

.. + *correlation functions with coincident fields*
+ *expectation values of normal ordered products of operators*
+ .... .

The general combination is

(4)
$$(p_{i_1}\cdot p_{j_1})^{r_1}...(p_{i_m}\cdot p_{j_m})^{r_m}(G(z_{i_1},z_{j_1}))^{r_1}...(G(z_{i_m},z_{j_m}))^{r_m}$$
$$r_1+....+r_m=n.$$

Suppose that i_1, j_1, i_2, j_2, ..., i_m, j_m are different. The combinatorial factor from the expansion of the correlation function is

(5)
$$\frac{1}{(r_1!)^2}\cdots\frac{1}{(r_m!)^2}r_1!...r_m! = \frac{1}{r_1!...r_m!}.$$

From the expansion of the exponential $exp\left(-\sum_{i<j}p_i^\mu p_j^\nu\langle 0|X_\mu(z_i)X(z_j)|0\rangle\right)$ is

(6)
$$\frac{1}{n!}\begin{pmatrix} n \\ r_1\ ...\ r_m\end{pmatrix} = \frac{1}{r_1!...r_m!}.$$

If some of the indices i_1, j_1, i_2, j_2, ...i_m j_m coincide, an alternate labelling $k_1,...,k_\ell$ of the expression is

(7)
$$(p_{k_1}\cdot p_{k_2})^{r_1}(p_{k_2}\cdot p_{k_3})^{r_2}...(p_{k_{i_0}}\cdot p_{k_{i_0+1}})^{r_{m_0}}...(p_{k_{\ell-1}}\cdot p_{k_\ell})^{r_m}G(z_{k_1},z_{k_2})^{r_1}...G(z_{k_{\ell-1}},z_{k_\ell})^{r_m}.$$

where p_{k_1} appears $r_{k_1} = r_{k_1,s_{11}} + ... + r_{r_1,s_{1,t_1}} = r_{k_1}$ times, ..., p_{k_ℓ} appears $r_{k_\ell} = r_{k_\ell,s_{\ell,1}} + ... + r_{k_\ell,s_{\ell,t_\ell}} = r_{k_\ell}$ times. The combinatorial factor from the expansion of the correlation function is

$$\begin{aligned}
(8)\qquad & \frac{1}{r_{k_1}!...r_{k_\ell}!}\begin{pmatrix} r_{k_1} \\ r_{k_1,s_{11}}...r_{k_1,s_{1,t_1}} \end{pmatrix}\cdots\begin{pmatrix} r_{k_\ell} \\ r_{k_\ell,s_{\ell,1}}...r_{k_\ell,s_{\ell,t_\ell}} \end{pmatrix} r_1!...r_m! \\
&= \frac{1}{r_{k_1,s_{11}}!...r_{k_1,s_{1,t_1}}!...r_{k_\ell,s_{\ell,1}}!...r_{k_\ell,s_{\ell,t_\ell}}!} r_1!...r_m! \\
&= \frac{1}{r_1!...r_m!}.
\end{aligned}$$

The expansion of the exponential again gives $\frac{1}{r_1!...r_m!}$. Therefore, the coefficients are equal, and expression for the correlation function is given by

$$(9)\qquad \prod_{i<j} exp\left(-p_i \cdot p_j G(z_i, z_j)\right)$$

with $G(z_i, z_j)\eta^{\mu\nu} = \langle 0|X^\mu(z_i), X^\nu(z_j)|0\rangle$. and not including the normalization of the Green function. The Green function for the free scalar field on the complex plan would satisfy

$$(10)\qquad \partial\bar{\partial}G(z_i, z_j) = \delta(z_i - z_j)$$

The solution is

$$(11)\qquad G(z_i, z_j) = \frac{1}{\pi}ln\ |z_i - z_j|$$

since

$$(12)\qquad \partial\bar{\partial}\ \frac{1}{2}(ln\ z + ln\ \bar{z}) = \left[\bar{\partial}\left(\frac{1}{z}\right) + \partial\left(\frac{1}{\bar{z}}\right)\right]$$

and

$$\begin{aligned}
(13)\qquad -\frac{i}{2}\int dzd\bar{z}\left[\bar{\partial}\left(\frac{1}{z}\right) + \partial\left(\frac{1}{\bar{z}}\right)\right] &= -\frac{i}{2}\frac{1}{2}\left[\int_C \frac{dz}{z} + \int_C \frac{d\bar{z}}{\bar{z}}\right]. \\
&= -\frac{i}{4}2 \cdot 2\pi i = \pi.
\end{aligned}$$

On a Riemann surface, two delta function sources are necessary for consistency with Stokes' theorem, and a summation over image charges under the action of a uniformizing Schottky group gives

(14)

$$G_{QS}(P,R) = \frac{1}{\pi}\Bigg[\sum_{\alpha} ln \ \frac{z_P - V_\alpha z_R}{z_Q - V_\alpha z_S}\frac{z_Q - V_\alpha z_S}{z_Q - V_\alpha z_R}$$

$$- \frac{1}{2\pi}\sum_{m,n=1}^{g} Re\ \{v_m(z_P) - v_m(z_Q)\}(Im\ \tau)^{-1}_{mn} Re\{v_n(z_R) - v(z_S)\}\Bigg]$$

$$v_n(z) = \sum_{\alpha}{}^{(n)} ln\left(\frac{z - V_\alpha \xi_{1n}}{z - V_\alpha \xi_{2n}}\right) \qquad v_n(z) - v_n(T_m z) + 2\pi i \tau_{mn},$$

where R and S represent the sources, τ is the period matrix, and $\{v_n,\ n = 1,,,.g\}$ is the set of integrals of canonical basis of homolorphic differentials and the sum over elements $\{V_\alpha\}$ over the free group $\langle T_1^{\pm 1}, ..., T_g^{\pm 1}\rangle$, $\Sigma^{(n)}$ does not include those products of generators that have $T_n^{\pm 1}$ as the right-most factor. The final expression

(15)
$$\prod_{i<j} e^{-\frac{\pi}{2} p_i \cdot p_j G(z_i, z_j)}$$

then would be required to obtain an integral of the form

(16)

$$f(z_1^0, z_3^0, z_3^0) \int_\Delta d^2 z_4 |z_4 - z_1^0|^{1 - \frac{p_1 \cdot p_4}{2}} \prod_{\alpha \neq I}{}' \left| \frac{z_4 - V_\alpha z_1^0}{z_4 - V_\alpha z_4} \frac{z_1^0 - V_\alpha z_4}{z_1^0 - V_\alpha z_1^0} \right|^{-\frac{p_1 \cdot p_4}{2}}$$

$$\times \prod_{m,n} exp\left[\frac{p_1 \cdot p_4}{8\pi} Re\{v_m(z_4) - v_m(z_1^0))(Im\ \tau)^{-1} Re(v_n(z_r) - v_n(z_1^0))\}\right]$$

$$\times\ (similar\ factors\ with\ z_1^0 \to z_2^0,\ z_3^0,\ p_1 \to p_2,\ p_3).$$

The four-point finite-genus amplitude, for example, then may be expressed in the Schottky group coordinate parameterization of moduli space by

(17)

$$
\begin{aligned}
&\int_{F_g} d^2z\ |z-z_1^0|^{-\frac{p_1\cdot p_4}{2}}|z-z_2^0|^{-\frac{p_2\cdot p_4}{2}}|z-z_3^0|^{-\frac{p_3\cdot p_4}{2}}\Phi(z,\bar z)\\
&= \int_0^{2\pi}\int_{|z-z_1^0|\le\Lambda} d\theta\ d|z-z_1^0|\ |z-z_1^0|^{1-\frac{p_1\cdot p_4}{2}}\sum_{\ell=0}^{\infty}\sum_{m=0}^{\infty}\frac{(z-z_1^0)^\ell(\bar z-\bar z_1^0)^m}{\ell!m!}\\
&\qquad\qquad \partial^\ell\bar\partial^m\{|z-z_2^0|^{-\frac{p_2\cdot p_4}{2}}|z-z_3^0|^{-\frac{p_3\cdot p_4}{2}}\Phi(z,\bar z)\}_{z=z_1^0}\\
&\quad + \int_0^{2\pi}\int_{|z-z_2^0|\le\Lambda} d\theta\ d|z-z_2^0|\ |z-z_2^0|^{1-\frac{p_2\cdot p_4}{2}}\sum_{\ell=0}^{\infty}\sum_{m=0}^{\infty}\frac{(z-z_2^0)^\ell(\bar z-\bar z_2^0)^m}{\ell!m!}\\
&\qquad\qquad \partial^\ell\bar\partial^m\{|z-z_1^0|^{-\frac{p_1\cdot p_4}{2}}|z-z_3^0|^{-\frac{p_3\cdot p_4}{2}}\Phi(z,\bar z)\}_{z=z_2^0}\\
&\quad + \int_0^{2\pi}\int_{|z-z_3^0|\le\Lambda} d\theta\ d|z-z_3^0|\ |z-z_3^0|^{1-\frac{p_3\cdot p_4}{2}}\sum_{\ell=0}^{\infty}\sum_{m=0}^{\infty}\frac{(z-z_3^0)^\ell(\bar z-\bar z_3^0)^m}{\ell!m!}\\
&\qquad\qquad \partial^\ell\bar\partial^m\{|z-z_1^0|^{-\frac{p_1\cdot p_4}{2}}|z-z_2^0|^{-\frac{p_2\cdot p_4}{2}}\Phi(z,\bar z)\}_{z=z_3^0}\\
&\quad + \ \mathit{finite}\\
&= \sum_n \frac{\Lambda^{-\frac{p_1\cdot p_4}{2}+2n+2}}{-\frac{p_1\cdot p_4}{2}+2n+2}\frac{1}{(n!)^2}(\partial\bar\partial)^n\{|z-z_2^0|^{-\frac{p_2\cdot p_4}{2}}|z-z_3^0|^{-\frac{p_3\cdot p_4}{2}}\Phi(z,\bar z)\}_{z=z_1^0}\\
&\quad + \sum_n \frac{\Lambda^{-\frac{p_2\cdot p_4}{2}+2n+2}}{-\frac{p_2\cdot p_4}{2}+2n+2}\frac{1}{(n!)^2}(\partial\bar\partial)^n\{|z-z_1^0|^{-\frac{p_1\cdot p_4}{2}}|z-z_3^0|^{-\frac{p_3\cdot p_4}{2}}\Phi(z,\bar z)\}_{z=z_2^0}\\
&\quad + \sum_n \frac{\Lambda^{-\frac{p_3\cdot p_4}{2}+2n+2}}{-\frac{p_3\cdot p_4}{2}+2n+2}\frac{1}{(n!)^2}(\partial\bar\partial)^n\{|z-z_1^0|^{-\frac{p_1\cdot p_4}{2}}|z-z_2^0|^{-\frac{p_3\cdot p_4}{2}}\Phi(z,\bar z)\}_{z=z_3^0}\\
&\quad + \ \mathit{finite}
\end{aligned}
$$

where the domains of the integrals are divided into $\{z||z-z_i^0|\le\Lambda\}\cup F_g/\{z||z-z_i^0|\le\Lambda\}$, $i=1,2,3$ in the Schottky covering surface $C\cup\infty/\cup_{n=1}^{g}(D_{T_n}\cup D_{T_n^{-1}})$, where $\Phi(z,\bar z)$ is a regular function. The poles in the amplitude occur at $s,\ t,\ u=8(n-1)$, beginning with the tachyon and progressing to excited levels. When $n=0$, the poles occur at $s,\ t,\ u=-8$, which would describe an intermediate tachyon resonance. Another linear dependence, such as $4n-12$, which would arise if the exponents included $-p_i\cdot p_4$ without the factor of $\frac{1}{2}$, yields intermediate states with squared center-of-mass energies equal to -12 and, after iteration of the scattering calculation, a set of resonances with arbitrarily large negative energies. Consequently, the factor of $\frac{\pi}{2}$ would be present in the factor (15), and the tachyon vertex operator actually must have the form $e^{i\sqrt{\frac{\pi}{2}}p\cdot X(z)}$. The spacings are determined by the tension of the string. Therefore, the energies can range from Planck to hadronic scales.

An N-point correlation function in superstring theory would be equal to

$$(18) \qquad \langle V_1 \,...\, V_N \rangle = \sum_{g=0} \kappa^{2g-2} \int_{sM_g} d\mu_{WP} \prod_{i=1}^{N} d^2 z_i \langle V_1(z_1) \,...\, V_N(z_N) \rangle$$

Expanding about a particle resonance

$$(19) \qquad \sum_{g=0} \kappa^{2g-2} \sum_{k=1}^{g-1} A_k \frac{1}{p^2 - m^2} A_{g-k}$$

It may be recalled that the upper bound for the amplitude at genus g is given by the maximum over the interior and the compactification divisor. The integral over Schottky group parameters over the interior, which is the complement of the neighbourhood of the divisor, is less than $\frac{B^g}{g!}$, where the quotient by $g!$ results from the restriction to the fundamental domain of the supermodular group [6]. Summing over the interior and the components of the compactification divisor describing two components, the bound for the sum over the amplitudes would be

$$(20) \quad \kappa^{2g-2} \left[\frac{B^g}{g!} + \sum_{\substack{g_1+g_2=g \\ 1 \leq g_1, g_2 \leq n-1}} \frac{B^{g_1}}{g_1!} \frac{B^{g_2}}{g_2!} \right] = \kappa^{2g-2} B^g \left[\frac{1}{g!} + \sum_{\substack{g_1+g_2=g \\ 1 \leq g_1, g_2 \leq n-1}} \frac{1}{g_1!} \frac{1}{g_2!} \right].$$

Let

$$(21) \qquad S_{2,g} = \sum_{\substack{g_1+g_2=g \\ 1 \leq g_1, g_2 \leq g-1}} \frac{1}{g_1!} \frac{1}{g_2!}$$

Since

$$(22) \qquad e^{x_1+x_2} = \sum_{g=0}^{\infty} \frac{(x_1+x_2)^g}{g!} = \sum_{g=0}^{\infty} \sum_{g_1+g_2=g} \frac{x_1^{g_1}}{g_1!} \frac{x_2^{g_2}}{g_2!},$$

$$(23) \qquad e^2 = \sum_{g=0}^{\infty} \frac{2^g}{g!} = \sum_{g=0}^{\infty} \sum_{g_1+g_2=g} \frac{1}{g_1!} \frac{1}{g_2!} = \sum_{g=0}^{\infty} S_{2,g}$$

and

$$(24) \qquad S_{2,g} = \frac{2^g}{g!}.$$

Then

$$(25) \qquad \sum_{\substack{g_1+g_2=g\\ 1\leq g_1,g_2\leq n-1}} \frac{1}{g_1!}\frac{1}{g_2!} = S_{2,g} - \frac{1}{g!} - \frac{1}{g!} = \frac{2^g-2}{g!} \qquad g \geq 1.$$

The magnitude of the amplitude is less than

$$(26) \qquad \kappa^{2g-2}\,|B|^g \left[\frac{1}{g!} + \frac{2^g-2}{g!}\right] = \kappa^{2g-2}\,|B|^g\,\frac{2^g-1}{g!}.$$

Then the summation of the factorizations over the particle resonance is bounded by

$$(27) \qquad \sum_{g=0}^{\infty} \kappa^{2g-2}\,|B|^g\,\frac{2^g-1}{g!}\,\frac{1}{p^2-m^2}.$$

The series converges for all values of the coupling, and the mass of the resonance remains invariant.

Infrared divergences resulting from propagation of tachyon states in bosonic string are cancelled by supersymmetry. Then there are massless states propagating along thin tubes created by pinching the separating cycle between the genus k and genus $g = k$ components of a surface of genus g. After these resonances are excited states of the string. The problem with this model is that the excited states of a Planck size string have Planck masses, which is too large for the elementary particle spectrum. The viability of a model of the particle spectrum based on superstring amplitudes would depend on existence of corrections to the mass of the resonance. It is evident from the above series that there is no mass renormalization in superstring perturbation theory. Only the residues, representing couplings to different channels in the scattering process [3], are altered by radiative corrections in the superstring perturbation series.

The nonperturbative contribution to the amplitude arises from the infinite-genus boundary components. The amplitude given by the integral over the set of effectively closed surfaces with zero harmonic measure would have the same form as finite-genus amplitudes and not affect the masses of resonances. However, there exists a class of surfaces beyond this category of O_G surfaces that represent the nonperturbative phenomena in the theory. A nonvanishing capacity of the ideal boundary yield a non-zero potential for the surface through the integration of the normal derivative of the harmonic measure [16]. The potential energy can be set equal to the energy of the infinite-genus boundary component. When there is no other dynamics, this energy would be proportional to a non-zero rest mass.

It is known also that the masses of leptons may be estimated through radiative corrections [20]. If the Planck strings are replaced by hadronic string with tensions that are multiplied by a factor $\left(\frac{\ell_P}{R_{hadr.}}\right)^2$, the masses of the excited states become more realistic. There is still a large difference between the masses of the hadrons and the leptons.

The formula for the mass of the electron in a quantum theory of radiating oscillators is precise, however, and it is indicative of the necessity of methods which are valid for the finite size. The equation for a radiating electron in an electromagnetic field is

$$m\ddot{x} - \frac{2e^2}{3c^3}\frac{d^3x}{dt^2} = eE \tag{28}$$

with $E = E_\omega cos(\omega t)$ has the solution

$$x(t) = -\frac{eE_\omega}{m\omega^2} cos\ \zeta_\omega cos(\omega t - \zeta_\omega) \tag{29}$$

where $tan\ \zeta_\omega = -\frac{\omega}{\omega_0}$. The Hamiltonian may be given by

$$H' = 4\pi c^2 \Omega^{-1} \sum_s \omega_s^{-2} \Big[(\sum_i e_i \beta_i sin\ \vartheta_{si} cos\zeta'_{si} sin(\Gamma_{si} + \zeta'_{si}))^2 + (\sum_i e_i cos\ \zeta'_{si} cos(\Gamma_{si} + \zeta'_{si}))^2 \Big] \tag{30}$$

and ϑ_{si} is the angle between the wave with index s and the velocity $\vec{\beta}_i$, while $\zeta'_{si} = arc\ tan\left(-\frac{\omega'_{si}}{\omega_0}\right)$, with $\omega_0 = \frac{3mc^3}{2e^2}$ and $\omega'_{si} = \omega_s(1 - \beta_i cos\ \vartheta_{si})(1-\beta_i^2)^{-\frac{1}{2}}$, and $\Gamma_{si} = \frac{\omega_s r_i cos\ \vartheta_{si}}{c} +$ [17]. For a single particle at rest, $\beta_i = 0$ and $\zeta'_{si} = \zeta_s$. It follows that

$$E_i^0 = \frac{4\pi c^2}{\omega} e_i^2 \sum_s \omega_s^{-2} cos^2 \zeta_s cos^2(\Gamma_{si} + \zeta_s). \tag{31}$$

Replacing the sums by an integral,

$$E_i^0 = \frac{4\pi c^2}{\omega} e_i^2 \frac{\omega}{2\pi^2 c^3} \int d\omega \left(1 + \frac{\omega^2}{\omega_0^2}\right)^{-1} cos^2(\Gamma_{si} + \zeta_s) \tag{32}$$

and setting $\langle cos^2(\Gamma_{si} + \zeta_s)\rangle = \frac{1}{2}$,

$$\langle E_i^0 \rangle = \frac{e_i^2}{\pi c} \int_0^\infty \left(1 + \frac{\omega^2}{\omega_0^2}\right)^{-1} d\omega = \frac{e_i^2 \omega_0}{2c} = \frac{e_i^2}{2c} \frac{3}{2} \frac{m_i c^2}{e_i^2} = \frac{3}{4} m_i c^2. \tag{33}$$

For the interaction between two charges, it may be noted that $cos(\Gamma_{si} + \zeta_s) cos(\Gamma_{sj} + \zeta_s) = \frac{1}{2} cos(\Gamma_{si} + \Gamma_{sj} + 2\zeta_s) + \frac{1}{2} cos(\Gamma_{si} - \Gamma_{sj})$ and, when $cos_{\vartheta_s} = 0$,

$\frac{1}{2}cos(\Gamma_{si}\Gamma_{sj}) = \frac{1}{2}cos\left(\frac{\omega_s(r_i-r_j)cos\ \vartheta_s}{c}\right) = \frac{1}{2}cos\left(\frac{\omega_s r_{ij}}{c}\right)$. Let $r_{ij} = r$, $\frac{\omega r}{c} = ux_0$ and $x_0 = \frac{r}{r_0}$, where $r_0 = \frac{2e^2}{3mc^2}$. Then

(34)

$$E^0_{ij} = \frac{e^2}{c}\frac{2}{\pi}\int_0^\infty du\frac{c}{r}x_0 sin(ux_0)(1+u^2)^{-1} = \frac{e^2}{r_0}\frac{2}{\pi}\int_0^\infty du sin(ux_0)(1+u^2)^{-1}.$$

The evaluation of the sum of oscillator contributions is the equivalent of a one-loop calculation of the self-energy of the electron. There also would be higher-order contributions in the diagrammatic expansion of quantum field theory. If the action may be included in a supersymmetric model, a nonrenormalization theorem may be used to render corrections to be vanishing. When supersymmetry is not present, the magnitude of any higher-order terms would have to be bounded.

A classical theory with form factors would begin with the replacement of the current by $e\int_{-\infty}^{\infty} v_\mu F(R^2)$ in the source term $\int j^\mu(x)A_\mu(x)d^4x$, where $F(R^2) = \frac{1}{4\pi^2}\int e^{-iK\cdot R}g(K^2)d^4K$, $g(K^2) = \frac{K_0^4}{4\pi^2(K_0^4+K^4)}$, and terms in the electron equation include $\left(M^2 + e^2\frac{C_0}{2r_0}\right)\dot{v}_\mu - \frac{2}{3}e^2(\ddot{v}_\mu + \dot{v}^2 v_\mu) + ...$, where $\frac{C_0}{r_0} = \int_{-\infty}^{\infty} H(\sigma^2)d\sigma$ and $H(R^2) = -4\pi\int\frac{g^2(K^2)}{K^2}e^{-iK\cdot R}d^4K$ [4][18]. The electromagnetic contribution to the energy is equal to $\frac{2}{3}\frac{e^2}{r_0}$ if

$$r_0\int_{-\infty}^{\infty} H(\sigma^2)d\sigma = \frac{4}{3}. \tag{35}$$

3. Integer Multiple Rule

The integer multiple rule has been observed amongst leptons and hadrons. It may be noted that the formula for the masses of dyons in N = 4 super-Yang-Mills theory [22] verify the integer multiple rule when either the electric or the magnetic charge is significantly larger [5]. The dyons represent nonperturbative states in string theory.

Nevertheless, shifts in the masses of hadronic states are expected to be derived from radiative corrections. By the formula for the bosonic string amplitude, it is evident that the poles occur at

$$\frac{p_i\cdot p_j}{2} = 2n+2 \tag{36}$$

or $s,\ t\ ,u = 8(n-1)$. Consequently, the squared center-of mass energy scales as the square root of n. The path integral formalism is developed to compute the scattering amplitudes of N massless bosons with vertex operators

$V_0 = \varepsilon_\mu(\partial x^\mu - ip_\nu\psi^\mu\psi^\nu)e^{ip\cdot x}$ and $V_{-\frac{1}{2}} = u^\alpha(p)S_\alpha(z)S_g^{-\frac{1}{2}}(z)e^{ip\cdot x}$ and $2n$ massless fermions with vertex operators $V_{-\frac{1}{2}} = u^\alpha(p)S_\alpha(z)S_g^{-\frac{1}{2}}(z)e^{ip\cdot x}$ with ghost charge $-\frac{1}{2}$ and $V_{\frac{1}{2}} = u^\alpha(p)S_\alpha(z)S_g^{\frac{1}{2}}e^{ip\cdot x}$ with ghost charge $\frac{1}{2}$, where S_α and $S_g^{\pm\frac{1}{2}}$ are spin fields [1]. The fermionic scattering amplitudes

(37)

$$A_N = \int dz\omega(z)d\xi\omega_\xi \prod_{j=3}^{N} dz_j\omega(z_j) \sum_{n=0}^{\infty} \frac{(\omega\xi)^{-p_1\cdot p_2+\vec{q}_{\alpha_1}\cdot\vec{q}_{\alpha_2}-\frac{1}{4}+n+1}}{n!}$$

$$\nabla^n_{z_i}[\Theta_R(z_1,z_2)F(z_1,z_2,z_j)]|_{z_1=z_2=z}$$

$$F(z_1,z_2,z_j) = \prod_{j=3}^{N} \Theta(z_i,z_j)^{\vec{q}_{\alpha_1}\cdot\vec{q}_j+\vec{p}_1\cdot\vec{p}_j+\frac{g_j}{2}}\Theta(z_2,z_j)^{\vec{q}_{\alpha_2}\cdot\vec{q}_j-\vec{p}_2\cdot\vec{p}_j+\frac{g_j}{2}}$$

$$\prod_{a=1}^{5}\Theta_m\left(q_{\alpha_i,a}\int_P^{z_1}\omega + q_{\alpha_2,a}\int_P^{z_2} + \sum_{j,a}\int_P^{z_j}\omega\right)$$

$$\left[\Theta_m\left(-\frac{1}{2}\int_P^{z_1}\omega - \frac{1}{2}\int_P^{z_2}\omega + \sum_j g_j\int_P^{z_j}\right)\omega\right]^{-1}K_{ij}$$

$$Kij = \prod_{2\le i\le j}[\Theta(z_i,z_j)]^{\vec{q}_i\cdot\vec{q}_j}{}_{-i}\cdot\vec{p}_j\cdot g_i\cdot g_j$$

where p_1 and p_2 are momenta of gauge bosons, $\vec{q}_{\alpha_1}$ and $\vec{q}_{\alpha_2}$ are fermion charges of the spin field, $\omega = \omega^A\frac{\partial\Theta(u^A)}{\partial u^A}\Big|_{u^i=0}$, with $\{\omega^A,\ A = 1,...,g\}$ is a canonical basis of holomorphic one-forms, ξ is the tangent vector to the geodesic connecting z_1 and z_2 in a normal coordinate expansion on the Riemann surface and g_i is the ghost charge, has poles at $p_1\cdot p_2 = \vec{q}_{\alpha_1}\cdot\vec{q}_{\alpha_2} - \frac{1}{4} + n + 1$ [1]. The squared center of mass energy is scaling linearly with the level number. Recalling that, in a center-of-mass frame, $s^2 = (p_1+p_2)^2$, it would equal

$$s = p_1^2 + p_2^2 + 2\vec{q}_{\alpha_1}\cdot\vec{q}_{\alpha_2} + 2\left(n+\frac{3}{4}\right). \tag{38}$$

Setting s at the resonances equal to M^2, where M is the mass of the intermediate state, when the vertex operators represent massless states with null momentum vectors,

$$M^2 = 2\vec{q}_{\alpha_1}\cdot\vec{q}_{\alpha_2} + 2\left(n+\frac{3}{4}\right). \tag{39}$$

The masses of the charged particles may be deduced from the dyon and Lorentz Formulae. It is feasible to demonstrate the integer multiple rule for more massive resonances. The use of the nonperturbative formula for the electron, however, would require introduce additional charges.

The intermediate vector boson in the fundamental interactions has integer spin to conserve angular momentum in the scattering of fermions with half-integer spin. By contrast, Regge amplitudes can have poles at the squared mass of baryons with half-integer spin. Baryons are massive compound states that have quantum numbers of the resonances on the Regge trajectories. The electron is an elementary particle that generally could be interpreted as an incoming or outgoing state and not as a resonance produced by the scattering.

Nevertheless, the formula (39) may be checked in this regard. The values of the inner products may be $\vec{q}_{\alpha_1} \cdot \vec{q}_{\alpha_2} = -\frac{3}{4}, \frac{1}{4}, \frac{5}{4}$. Then, when $n = 0$ and the spin field scalar product is $\frac{1}{4}$,

$$M^2_{n=0} = 2 \cdot \frac{1}{4} + 2 \cdot \frac{3}{4} = 2 \tag{40}$$

The ratio of the fermion contribution to the entire expression is $\frac{3}{2} : 2 = \frac{3}{4}$, which is identical to that of the Lorentz formula for the electromagnetic contribution to the mass of the electron. Since the equality (40) describes the squared mass instead of the mass, an alternative mechanism for describing a spin-$\frac{1}{2}$ intermediate state would be necessary. The existence of the poles in the scattering amplitude reflects the quadratic form of the propagator for integer spin particles. When the intermediate state has half-integer spin, it must be modified to be linear in $\frac{1}{p-m}$. Since

$$\frac{1}{p^2 - m^2} = \frac{1}{p+m} \frac{1}{p-m}, \tag{41}$$

it follows that if singularity in the scattering amplitude occurs at $p = m$, the pole in the fermion propagator would arise if $(p-m)\psi = 0$, where ψ is the spin-$\frac{1}{2}$ field, and the product would be $\frac{1}{2m}\frac{1}{p-m}$. The factor $(\omega\xi)^{-p_1 \cdot p_2 + \vec{q}_{\alpha_1} \cdot \vec{q}_{\alpha_2} - \frac{1}{4} + n + 1}$ in the integrand of the superstring amplitude arises from the representation of theta functions producing the expectation value of the product of pairs of fermion vertex operators, $V_{-\frac{1}{2}}$ and $V_{\frac{1}{2}}$, and the poles in the scattering amplitude may be found through an expansion in a Taylor series. The poles of the fermion propagator would be expected to arise from a factor with an exponent with a linear function of the momentum. The production of an intermediate spin-$\frac{1}{2}$ particle, however, requires one of the fermionic incoming particles to be replaced by a bosonic particle, with the effect of leaving an unpaired fermion in the expectation value. The contraction of factors in V_0 and $V_{\pm\frac{1}{2}}$ would include that of $e^{ip\cdot x}$ and ∂x_μ, which yields after integration over the Riemann surface, yields, a linear function of the momentum. It would also introduce only a single fermion charge of the spin field q_α

Setting this linear function equal to the mass at the pole of the amplitude then would yield the equation, since p^2 gives 2ν, half of the rest of the exponent for p would give

$$M^{spin-\frac{1}{2}} = \frac{1}{2}|\vec{q}_\alpha| + \left(n + \frac{3}{4}\right). \tag{42}$$

for a fermionic intermediate state. Since $|q_\alpha| = \frac{1}{2}$, when $n = 0$,

$$M^{spin-\frac{1}{2}}_{n=0} = \frac{1}{2}\cdot\frac{1}{2} + \frac{3}{4} = 1 \tag{43}$$

and the contribution of the matter fields would be $\frac{3}{4}$ of the mass of the spin-$\frac{1}{2}$ particle since the remainder would result from the spin field of the superstring.

The ratio of the masses of the γ branch of the meson and baryon spectrum to the pion mass satisfies an integer multiple rule [2][12] [15][19]. Fractional powers may result from a variety of potentials in the modeling of hadronic spectra [14]. Since the energy levels are defined for the scattering of two string states in a fixed channel, it would be best to consider the integer spacing for the squares of masses amongst the corrections to the fundamental mass spectrum within each particle multiplet. The Gell-Mann-Okubo mass rule is known to be linear for fermions and quadratic for mesons [13] [21], which describe bound states of two quarks, and therefore, a resonance resulting from the interaction of two strings.

Given a collection of pion bubbles in a meson with a mass larger by a factor of υ, the poles in the string amplitudes would yield corrections δM for each pion that are proportional to $\frac{1}{\upsilon}\sqrt{n}$. Given k excited energy levels of the open string between the quark and anti-quark in each pion, there would be $k\upsilon$ excited states of the meson.

4. **Conclusion**

The masses of the elementary particles are known to satisfy an integer multiple rule. It has been determined that this rule follows from the masses of nonperturbative string states which are dyons in N = 4 super-Yang-Mills theory. The perturbative string amplitudes have poles at values of s, t and u that increase linearly with the level number of the excited state of the intermediate resonance. It follows that this radiative process then yield quadratic corrections to a fundamental energy spectrum. The problem of the linear proportionality of masses with the level number may be resolved for half-integer spin particles by transferring the pole of the resonance to the factor $\frac{1}{p-m}$ and integer-spin particles through pion bubbles since the ratio of the mesons in the γ branch. The validity of the pion bubble model for the γ branch follows from the decay $\pi^0 \to 2\gamma$ for each pion. The Gell-Mann-Okubo mass rule is conjectured to be the primary method for predicting masses differences in the hadron multiplets.

Phenomenologically realistic values of the masses would be derived from the hadronic string rather than the superstring. Without supersymmetry, the infrared divergences resulting from tachyon states must be removed. There exists a method for ensuring finiteness of the string theory having only modular invariance and breaking of supersymmetry [9] through a matching of states in different sectors. The tachyon then would not generate the problems for such a model that occur for the bosonic string.

Given the meson resonances in the γ branch which have masses that are integer multiples of m_{π^0} or m_{π^-}, excited states resulting from the quantized vibrational modes of the open string connecting the quark and the anti-quark that differ from the mass of the resonance M by δM, where $\delta M \propto \frac{1}{v}\sqrt{n}$. There will be kv energy levels when the open string has k excited energy levels.

The variation from this rule for the ν branch is indicative of an extra contribution to the masses, which may be related to the infinite-genus boundary component that is found to be sufficient to explain a nonperturbative contribution to the η' mass [7].

Lepton masses would not be predicted by the current string models. A recent study of the generation of lepton masses is based on the division algebra modules in the spinor space of the standard model [8]. The infinite-genus boundary component is not required. However, the interaction between quarks and leptons in the weak interactions is indicative of a connection between the ideal boundaries of infinite-genus surfaces and the ν branch of the meson and baryon resonances. The electromagnetic contribution to the mass of a charged particle is given by the Lorentz law which depends on its radius. The formula for the energies of a set of elementary particle states therefore would include the combination of the integer multiple rule and a Lorentz term that represents symmetry breaking.

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